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STRUCTURAL PROPERTIES OF COMPACT GROUPS WITH MEASURE-THEORETIC APPLICATIONS

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ABSTRACT

Every compact group is Baire isomorphic to a product of compact metric spaces; the isomorphism takes the Haar measure on the group to a direct product measure. This topological connection between compact groups and products of compact metric spaces provides a unified treatment for (Baire) measures on compact groups and for measures on topological products of metric spaces.

1. Introduction and review of terminology

There are several articles on the realization of σ-homomorphisms by point maps and the pertinent bibliography has been cited in [10]. As basic papers we consider the classical publication [17] of von Neumann, [14] of Maharam and [1] of Choksi. Further studies are contained in Choksi [2], Choksi and Simha [6], Maharam [15] and Graf [9]. The study [4] and particularly Fremlin [7] are also noteworthy.

In 1979 Choksi and Fremlin developed a theory [5] which was designed to study how many, i.e. non-isomorphic, completion regular measures can exist on a product of compact metric spaces. The results of [5] not involving additional set-theoretic assumptions remain valid for arbitrary compact groups [10].

The present paper again returns to the topic of measurable transformations on topological groups and proves that every compact group is Baire isomorphic to some product of compact metric spaces of the same topological weight (for

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the precise statement see Theorem 2.3). Our method is, essentially, a slight refinement of that used in [10]. Various extensions and refinements (with short proofs) of known theorems in topological measure theory are deduced. The results obtained complement the discussion presented in [10] and yield a theory for Baire measures on compact groups which is, in a certain sense, a complete analogue of that for Baire measures on products of compact metric spaces.

All measure spaces (Z, Σ, λ), simply denoted by (Z, λ), are assumed to be finite. The completion of Σ with respect to λ is denoted by $\bar{\Sigma}^{\lambda}$.

Let $(Z_i, \mathcal{B}_i, \lambda_i)$, $i = 1, 2$ be measure spaces. A map $f : Z_1 \to Z_2$ is called $\mathcal{B}_2 - \mathcal{B}_1$ measurable if $f^{-1}B \in \mathcal{B}_1$ for all $B \in \mathcal{B}_2$. For an arbitrary compact (Hausdorff) space X, let $\mathcal{B}_X$, respectively $\mathcal{B}_X^0$, denote its Borel respectively Baire σ-algebra. A Baire measure μ on X is called completion regular if every Borel set in X is μ-measurable. In the case when $X = G$ is a compact group, λ_G will denote the (normalized) Haar measure on G. Finally, in that case R_G will be the set of closed normal subgroups of G. (The reader is referred to [11], [12] for further information about measures on compact groups.)

Next, let X, Y be the compact spaces and μ, ν Baire measures on X, respectively Y. A map $g : X \to Y$ is called Baire (respectively completion Baire) measurable iff it is $\mathcal{B}_Y^0 - \mathcal{B}_X^0$ (respectively $\overline{\mathcal{B}_Y^0}^{\nu} - \overline{\mathcal{B}_X^0}^{\mu}$) measurable. A (completion) Baire measurable bijection g is said to be a (completion) Baire isomorphism iff g^{-1} is also (completion) Baire measurable. If such a bijection exists, then X and Y, respectively (X, μ), (Y, ν)—or just μ, ν—are said to be (completion) Baire isomorphic.

2. The Baire isomorphism

Throughout this paper G is a compact topological group of uncountable weight $w(G) = \alpha$. For $H \in R_G$, P_H will denote the canonical projection from G to G/H. We need some auxiliary notation.

If X is a compact space, (Y, Σ, μ) any measure space and h a mapping from Y to X such that $h^{-1}C \in \Sigma$ for every $C \in \mathcal{B}_X^0$, then $\bar{h}(\mu)$ will denote the Baire measure on X defined by: $h(\mu)(D) = \mu(h^{-1}D)$, for $D \in \mathcal{B}_X^0$.

In the sequel, we shall identify a cardinal γ with the initial ordinal of cardinal γ. Also, when no ambiguity arises, we shall identify a Radon measure with its restriction to the Baire σ-algebra.

Before stating our main result, the following lemma concerning Baire cross-sections is proven. It is a refinement of lemma 5.6 in [10].

LEMMA 2.1: *Let $H \in R_G$. If H is Lie, then there exists a Baire set S in G such that* (a) $G = S.H$, (b) $p_H^{-1}\{\dot{x}\} \cap S$ *is a single point set* $\{r(\dot{x})\}$ *for all* $\dot{x} \in G/H$ *(where* $\dot{x} = x.H$, $x \in G$), (c) *the bijection* $q : G/H \times H \to G$ *defined by:* $q(\dot{x}, t) = r(\dot{x}).t$ *is a Baire isomorphism from* $G/H \times H$ *onto* G *such that* $q(\lambda_{G/H \times H}) = \lambda_G$.

Proof (included for completeness): By theorem 1 in section 5.4 of [16], there exist a compact neighborhood Q_1 of id_G and a compact $F_1 \subset Q_1$ such that (i) $Q_1 = F_1.H$, (ii) $p_H^{-1}\{\dot{x}\} \cap F_1$ is a single point, for $\dot{x} \in p_H Q_1$. Let Q_2 be an open Baire set in G such that $Q_2 \subset Q_1$. Since p_H is an open surjection, $p_H Q_2$ is a Baire subset of G/H (see e.g. [13], lemma 1.6, p. 275). Thus the open set $Q = p_H^{-1} p_H Q_2 = Q_2.H$ must be Baire in G.

Now if we set $F = \{x \in F_1 : p_H x \in p_H Q_2\}$, then we clearly have $Q = F.H$ and the rest of the proof proceeds exactly as that of lemma 5.6 in [10]. ∎

Note: Lemma 2.1 was suggested by a technique used by Choksi in [2], Lemma 6.

In the sequel, for any family $\{\mu_i\}$ of Radon probability measures, $\bigotimes_i \mu_i$ will denote, as usual, the radon product measure of the μ_i. The next statement, which is immediate from arguments used in [10], is modelled on that of Furstenberg's structure theorem (see e.g. [8]).

LEMMA 2.2: *There exists a family* $(H_\gamma)_{\gamma<\alpha}$ *of groups in* R_G *with* $\bigcap_{\gamma<\alpha} H_\gamma = \{\mathrm{id}_G\}$ *such that, for every* $\gamma < \alpha$,

(i_1) $H_\delta \subset H_\gamma$ *for* $\alpha \geq \delta \geq \gamma$,

(i_2) $H_\gamma / H_{\gamma+1}$ *is Lie and*

(i_3) $H_\gamma = \bigcap_{\beta<\gamma} H_\beta$ *if* γ *is a limit ordeal.*

Proof: There is a directed set $\Gamma = \{F_j,\ j \in J\}$ of groups in R_G, with the G/F_j Lie, of cardinal α such that $\bigcap_{j \in J} F_j = \{\mathrm{id}_G\}$[16]. The family $(H_\gamma)_{\gamma<\alpha}$ will be defined by transfinite induction on γ. Enumerating Γ as $\Gamma = \{F_\xi,\ \xi < \alpha\}$ and taking $H_0 = F_0$, we set $H_\gamma = H_\delta \cap F_\gamma$ if $\gamma = \delta + 1$ for some $\delta < \alpha$ and $H_\gamma = \bigcap_{\delta<\alpha} H_\delta$ otherwise.

Then, by construction, the H_γ satisfy the conditions i_1, i_2 and i_3. ∎

The following theorem establishes an intimate connection between compact groups and products of compact metric spaces.

THEOREM 2.3: *There exist a family $(\mu_\gamma)_{\gamma<\alpha}$ of Radon probability measures, each μ_γ supported on some compact metric space X_γ with at least two points and a Baire isomorphism q_G from $X_G := \prod_{\gamma<\alpha} X_\gamma$ onto G such that $g_G(\bigotimes_{\gamma<\alpha}\mu_\gamma) = \lambda_G$.*

Proof: (It proceeds exactly as that of the first part of corollary 5.11 of [10] except that lemma 2.1 replaces lemma 5.7 of [10]; the main steps of the proof are given only for completeness.)

Set $X_0 := G/H_1$ and $X_\gamma := H_\gamma/H_{\gamma+1}$ for $\gamma > 0$. By induction on $\gamma < \alpha$, using arguments involving projective limits of compact spaces, Lemma 2.1 and Lemma 2.2, for every $\gamma < \alpha$ we find a Baire isomorphism q_γ from $Y_\gamma = \prod_{\delta<\gamma} X_\delta$ onto G/H_γ such that $q_\varepsilon \circ r_{\zeta,\varepsilon} = p_{H_\zeta,H_\varepsilon} \circ q_\zeta$ for $\varepsilon < \zeta < \alpha$, where $p_{A,B}$ (resp. $r_{\zeta,\varepsilon}$) is the canonical projection of G/A to G/B (resp. Y_ζ to Y_ε). [If γ is a successor ordinal, say $\gamma = \delta + 1$, then Lemma 2.1 yields a Baire isomorphism $w_\gamma : G/H_\delta \times X_\delta \to G/H_\gamma$. We take $q_\gamma = w_\gamma \circ (q_\delta \times I_{X_\delta}) : Y_\gamma \to G/H_\gamma$, where $q_\delta \times I_{X_\delta}$ denotes the mapping: $Y_\delta \times X_\delta \to G/H_\delta \times X_\delta : (x,y) \to (q_\delta(x), y)$. In the case when γ is a limit ordinal we take q_γ to be the unique mapping satisfying: $q_\delta \circ r_{\gamma,\delta} = p_{H_\gamma,H_\delta} \circ q_\gamma$, $\delta < \gamma < \alpha$.]

Again there is a unique mapping $q_G : \prod_{\gamma<\alpha} X_\gamma \to G$ ($\cong \mathrm{proj}_\gamma \lim G/H_\gamma$) such that

$$p_{H_\gamma} \circ q_G = q_\gamma \circ p_\gamma, \quad \gamma < \alpha \tag{2.4}$$

(where p_γ denotes the canonical projection from $\prod_{\delta<\alpha} X_\delta$ onto $\prod_{\delta<\gamma} X_\delta$). Then q_G, as defined, is 1–1 by construction. Since $\prod_{\gamma<\alpha} X_\gamma$ is a cartesian product, it is immediate from (2.4) that q_G is surjective and in view of (2.4) and Lemma 2.1, is a Baire isomorphism satisfying the required conditions. This completes the proof of the theorem. ∎

3. Conseqences of Theorem 2.3

Theorem 2.3 makes possible the discussion of results about products of compact metric spaces in the setting of topological groups; in particular, combining arguments due to Choksi [1] with Theorem 2.3, the following alternative and totally different proof of theorem 1 in [2], independent of the deep induction arguments used there, is deduced.

THEOREM 3.1: ([2], Theorem 1) *Let μ be a finite Baire measure on G. Then every automorphism Φ of the measure algebra Ω_μ of $(G, \mathcal{B}_G^0, \mu)$ is induced by an invertible completion Baire point transformation T of G.*

Proof: By Theorem 2.3, there exist $X = X_G$ and $q = q_G$ as in Theorem 2.3. Consider the Baire measure $\nu = q^{-1}(\mu)$ on X and the bijection f from Ω_μ to the measure algebra Ω_ν of ν defined by: $f[A]_\mu := q^{-1}[A]_\nu$, where for a ξ-measurable set M, $[M]_\xi$ denotes equivalence modulo ξ-negligible sets.

Then clearly, the mapping $\Psi = f \circ \Phi \circ f^{-1}$ is an automorphism of Ω_ν which, in view of [1], is induced by an invertible completion Biare point transformation $R : X \to X$. Then the mapping $T = q \circ R \circ q^{-1} : G \to G$ has the required properties. ∎

Similarly, combining theorem 1 of [5] with Theorem 2.3, one easily gets

THEOREM 3.2: ([10], theorem 4.10) *If λ, μ are Radon probability measures on G and if there exists a measure preserving isomorphism of the measure algebras Ω_λ, Ω_μ of λ, respectively μ, then λ, μ are completion Baire isomorphic.*

Concluding remarks 3.3: We conclude this note by recording some more measure-theoretic applications of Theorem 2.3.

1. Let X_G, q_G be as in Theorem 2.3. First we observe that X_G is a compact group, because it is the direct sum of the (groups) X_γ. There is a natural way to associate a Baire measure $\mu = q_G(\nu)$ on G to any Baire measure ν on X_G and we clearly have that μ is homogeneous iff ν is homogeneous (of the same Maharam type). Moreover, μ is the (normalized) Haar measure on G if ν is a direct product measure on X_G. Thus, the Haar measure on any compact group is homogeneous, of Maharam type $w(G)$.

Here arises the natural question whether the completion regularity of μ implies the completion regularity of ν; the proof of theorem 5.8 in [10] does not hold.

2. In view of Theorem 2.3, it is easy to obtain results much more general than Theorem 3.2 applying to automorphisms between measure algebras. For example, two remarkable results, namely 2.22 and 2.23 in Fremlin [7], remain valid in the area of compact groups.

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